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Multiple and inverse topplings in the Abelian Sandpile Model

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Abstract. The Abelian Sandpile Model is a cellular automaton whose discrete dynamics reaches an out-of-equilibrium steady state resembling avalanches in piles of sand. The fundamental moves defining the dynamics are encoded by the *toppling rules*. The transition monoid corresponding to this dynamics in the set of stable configurations is *abelian*, a property which seems at the basis of our understanding of the model. By including also *antitoppling rules*, we introduce and investigate a larger monoid, which is *not abelian* anymore. We prove a number of algebraic properties of this monoid, and describe their practical implications on the emerging structures of the model.

1 Introduction

In 1987 Bak, Tang and Wiesenfeld [1] proposed a simple cellular automaton model of sandpile growth (since now BTW model) as an example of self-organized criticality (SOC). These systems should be characterized by a dynamics which sometimes, in an apparently unpredictable way, shows bursts of activity, avalanches in the sandpile, which eventually drive the system into an out-of-equilibrium steady state. The state is a steady state, in the sense that overall properties stay unchanged during time evolution, and out-of-equilibrium because these systems are open and dissipative and, therefore, they require input from outside at a constant rate to balance the dissipation, and present a spatial distribution of current. These steady states are critical as they exhibit long-range correlations with power-law decay.

Consider a rectangular portion of a square lattice of size $L_x \times L_y$, and let $n = L_x L_y$. To each site $i = (x, y)$ is associated a non-negative integer *height variable* (sometimes called *mass*) z_i. A *configuration* is the collection of all heights and will be denoted as a bold letter: $\mathbf{z} = \{z_i\}_{i=1}^{n}$. If all z_i's are smaller than 4, the configuration is *stable*. Otherwise, a procedure called *relaxation* gives univocally a new configuration $\mathcal{R}(\mathbf{z})$. The relaxation is performed through a sequence of elementary moves, called *topplings*: if $z_{(x,y)} \geq 4$ (in this case we say that the site (x, y) is *unstable*), then we decrease

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$z_{(x,y)}$ by 4, and increase by 1 all the values at neighbouring sites, $z_{(x\pm1,y)}$ and $z_{(x,y\pm1)}$ (nothing is done at those coordinates $(x\pm1,y)$, $(x,y\pm1)$ falling outside the grid). Topplings are performed at the various unstable sites, until when the configuration obtained is stable.

It is remarkable, and important, that the final result does not depend on the order in which the topplings are performed. In particular, if i and j are unstable sites in the configuration $\mathbf{z}$, then j is still unstable after a toppling has been performed at i (and viceversa). Indeed, all the valid sequences of topplings differ only for their ordering (although not all the permutations of a valid sequence are valid sequences).

Note that, if we call $|\mathbf{z}| = \sum_{i=1}^{n} z_i$ the *total mass* of the configuration, the mass decreases for topplings occurring at the boundary of the grid, and is conserved for topplings occurring in the bulk.

Call a_i the operator which maps a stable configuration to a new one by the addition of a grain of sand at the site i, followed, if it becomes unstable, by a relaxation. We have n operators a_i's, acting as a transition monoid on the set of stable configurations. As first pointed out by Dhar [2], these operators *commute*, i.e., for all $\mathbf{z}$, the configurations $a_i a_j \mathbf{z}$ and $a_j a_i \mathbf{z}$ coincide. As a result, the transition monoid is *abelian.* Under the Markov Chain obtained by acting with the a_i's, chosen at random, the sandpile is ergodic in a subset of all possible stable configurations. According to the general notion used in the analysis of Markov Chains, these configurations are called *recurrent*, and they are characterized by the absence of *forbidden sub-configurations* (FSC). All recurrent configurations occur with uniform probabilities in the steady state of this dynamics. A stable configuration which is not recurrent, i.e. that has zero steady-state probability, is called *transient.* As we will recall later on, the notions of recurrent and transient configurations can also be posed in purely algebraic terms, with no need to refer to probabilistic notions.

In the steady state the fluxes of sand added to and flowing out of the system must balance. This balancing occurs however in a non-trivial way, as the distribution of the size of the *avalanches*, i.e., the number of topplings required in the relaxation process at a given step of the Markov Chain, has an algebraic tail, although the precise exponent is still unknown [3,4,5,6].

The algebraic behaviour of the distribution of avalanche sizes suggests a relation with a state at criticality. Indeed, there is an equivalence between the sandpile model and the $q \to 0$ limit of the q-state Potts model, at criticality, through a common combinatorial description in terms of (rooted) spanning trees [7]. Therefore this sandpile critical behaviour is in the same universality class of a conformal field theory (CFT) with central charge $c = -2$, a relation which has allowed to obtain many exact results on the critical exponents associated to quantities that, differently from avalanche sizes, are more easily related to fields in the CFT. For example, the correlation of height variables at two sites decays only algebraically with the Euclidean distance between them [8], the distribution of the heights has corrections, depending on the Euclidean distance from the boundary, that decay only algebraically, and, consistently with the fact that the CFT at $c = -2$ is logarithmic, certain two-point correlation functions, or one-point boundary correlation functions, show the logarithmic corrections to power-law decays that one expects from the presence of Jordan blocks in the representation of the operator L_0 of the Virasoro Algebra (see [9,10,11,12,13,14,15] for a collection of both rigorous and non-rigorous theoretical results in this direction).

The dynamics of the sandpile, historically introduced first in [1] for the square lattice, is so simple and natural that it has been soon generalized to an arbitrary graph [2], an extension that allows to elucidate the main algebraic elements of the theory, and also independently studied by combinatorialists, under the name of *Chip Firing Game* [16,17,18,19,20,21], in relation to the Tutte polynomial of the graph (not surprisingly, as the Tutte polynomial of a graph is a reformulation of the Potts Model

partition function, the adjacency between variables being encoded by the graph, see e.g. [22]). In particular, a series of papers [19,20,21] shows how the generating function of recurrent configurations, weighted according to their total mass, gives access to larger class of enumerating problems, in particular, of spanning connected subgraphs and spanning forests.

Also our personal interest was induced by the relation of the sandpile model on an arbitrary graph and the $q \to 0$ Potts model, because we have introduced a novel representation in terms of fermionic variables of a model of random forests [23,24], that generalizes the Kirchhoff Matrix-Tree Theorem, corresponding to the case of spanning trees.

More recently, we have been strongly impressed by some general results which have been obtained for dynamics where all the sand is deterministically injected in a limited region [25,26], producing complex and beautiful patterns which display allometry. This has led to a long-term research project, of which some first results are reported in [27]. We believe that an important ingredient to better understand the emergence of these patterns, even for the ordinary Abelian Sandpile described above, is to understand two distinct, and apparently unrelated, generalisations of the model. The first more general toppling rules concern *multitopplings*, i.e. topplings associated to clusters of sites [28,29]. The second one concerns *antitopplings*, where instead of adding sand, and then relaxing, the sand is removed, and then an inverse relaxation is performed. An analogous of this latter elementary move had already been considered in [30], with the aim of determine theoretically certain critical exponents associated to the model in two dimensions, and, in the same variant considered here, has been studied later on, mostly numerically, in [31].

In this paper we will mostly present theoretical results for the transition monoid associated to the sandpile in which both topplings and antitopplings are allowed (remark that this monoid is no more abelian). These results are in the form of non-trivial algebraic relations (see e.g. Theorem 1 later on). Along the paper we will see how one of the theorems in the theory of antitopplings (Theorem 2) has a relatively simple proof only in the framework of the theory of multitopplings, this explaining the need of treating the two generalisations simultaneously.

The paper is organised as follows. Section 2, by far the longest of the paper, gives in Section 2.1 a first reminder of the properties of the Abelian Sandpile Model, then summarizes in Section 2.2 a collection of new theoretical results, finally in Section 2.3 more subtle aspects of the Abelian Sandpile Model are reviewed in detail. Due to a natural involution symmetry between toppling and antitopplings, the latter are introduced from the very beginning, even in the review parts. Conversely, the introduction and study of multitopplings is postponed to Section 3, where Theorem 2, presented in Section 2.2, is proven. Section 4 presents numerical examples (on the BTW setting) that motivate the study of several sandpile dynamics involving simultaneously topplings and antitopplings.

2 Algebraic formalism

The reader interested in the general theory of Abelian Sandpiles will find in [32] a beautiful review. An updated version is in [33], and other details are reported in [34]. In Section 2.1 we shall first report the basic facts that we need for our new developments. Note that, differently from the treatments in the references above, we introduce antitopplings, as well as several other “anti-” quantities, from the very beginning, and we choose a notation that highlights the role of a simple natural involution, that exchanges ordinary and the conjugated “anti-” quantities.

Facts already described in [32] are reported here for both classes of quantities. Indeed, because of the involution symmetry, the action of the conjugated operators alone cannot lead to any special novelty. Nonetheless, non-trivial aspects will arise when ordinary and conjugated operators *simultaneously* enter the game. The first simple properties in this wider context are discussed along Section 2.1, while the most remarkable facts are anticipated in Section 2.2, and proven in the body of the paper. Section 2.3 presents further aspects of the theory as developed in [32], that are not required for the understating of the statements of our results, but are useful in their proofs.

2.1 Definitions and basic facts

Let n be an integer, the *size* of the system. It is often useful to think at the system as a graph with n sites, and the set of toppling rules in terms of the adjacency structure of the graph, thus we will call *sites* the indices $i \in [n] \equiv \{1, \dots, n\}$. Consider vectors $\mathbf{z} \in \mathbb{Z}^n$, where we will use the partial ordering $\preceq$ such that $\mathbf{u} \preceq \mathbf{v}$ if $u_i \leq v_i$ for each $i \in [n]$. We will define the *positive cone* Ω as the subset of $\mathbf{w} \in \mathbb{Z}^n$ such that $\mathbf{w} \succeq \mathbf{0}$, where $\mathbf{0}$ has vanishing entries for all i.

An abelian sandpile $\mathcal{A} = \mathcal{A}(\Delta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$ is identified by a triple $(\Delta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$, that we now describe. The vectors $\overline{\mathbf{z}}$ and $\underline{\mathbf{z}}$ are the collection of *upper-* and *lower-thresholds*, $\{\overline{z}_i\}$ and $\{\underline{z}_i\}$ respectively, and are constrained to the condition $\overline{z}_i - \underline{z}_i > 0$ for all i. Define the spaces

$$S_+ = \{\mathbf{z} \in \mathbb{Z}^n \,|\, \mathbf{z} \succeq \underline{\mathbf{z}}\} = \bigotimes_{i=1}^{n} \{\underline{z}_i, \underline{z}_i + 1, \dots\}\,; \tag{1}$$

$$S_- = \{\mathbf{z} \in \mathbb{Z}^n \,|\, \mathbf{z} \preceq \overline{\mathbf{z}}\} = \bigotimes_{i=1}^{n} \{\dots, \overline{z}_i - 1, \overline{z}_i\}\,; \tag{2}$$

$$S = \{\mathbf{z} \in \mathbb{Z}^n \,|\, \underline{\mathbf{z}} \preceq \mathbf{z} \preceq \overline{\mathbf{z}}\} = \bigotimes_{i=1}^{n} \{\underline{z}_i, \dots, \overline{z}_i\} = S_+ \cap S_-\,. \tag{3}$$

Thus S_+ and $-S_-$ are translations of Ω, while S is a multidimensional interval, and has finite cardinality.

We also have a $n \times n$ *toppling matrix* Δ, with integer entries, that should satisfy $\overline{z}_i - \underline{z}_i + 1 \geq \Delta_{ii} > 0$, and $\Delta_{ij} \leq 0$ for $i \neq j$. We say that the sandpile is *tight* if $\overline{z}_i - \underline{z}_i + 1 = \Delta_{ii}$ for all i. Define the spaces

$$S'_+ = \bigotimes_{i=1}^{n} \{\overline{z}_i - \Delta_{ii} + 1, \overline{z}_i - \Delta_{ii} + 2, \dots\}\,; \tag{4}$$

$$S'_- = \bigotimes_{i=1}^{n} \{\dots, \underline{z}_i + \Delta_{ii} - 2, \underline{z}_i + \Delta_{ii} - 1\}\,. \tag{5}$$

We have $S_+ \supseteq S'_+$, and $S_- \supseteq S'_-$ in general, while $S_+ \equiv S'_+$, and $S_- \equiv S'_-$, if and only if the sandpile is tight.

We further require *dissipativity*, that is $b_i^- = \sum_j \Delta_{ij} \geq 0$. As seen below, b_i^- is the amount of mass that *leaves* the system after a toppling at i. The requirement that the toppling matrix is *irreducible*[1] ensures that the avalanches are finite (and that $\det \Delta > 0$). For future utility, we also define $b_j^+ = \sum_i \Delta_{ij}$, which is the difference between the amount of mass that can leave the site j in a toppling and the mass that can be added if all other sites would make a toppling. A site j where $b_j^+ < 0$ is said to

[1] This means that for every j_0 there exists a sequence $(j_0, j_1, \dots, j_\ell)$ such that $\Delta_{j_a\, j_{a+1}} < 0$ for all $0 \leq a < \ell$, and $b_{j_\ell}^- > 0$.

be *greedy* or *selfish*. Clearly $\sum_i b_i^- = \sum_i b_i^+$, so also the b_i^+'s are, on average, higher than zero, however positivity on the b_i^+'s is not implied directly by the positivity of the b_i^-'s. We require the absence of greedy sites as an extra condition, whose utility will be clear only in the following. A sandpile is said to be *unoriented* if $\varDelta = \varDelta^{\mathrm{T}}$ (and thus $\mathbf{b}^- = \mathbf{b}^+$).

The above conditions complete the list of constraints characterizing valid triples $(\varDelta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$. The special case of the BTW sandpile corresponds to $\varDelta$ being the discretised Laplacian on the square lattice, $\varDelta_{ii} = 4$, $\varDelta_{ij} = -1$ if $d(i,j) = 1$ and $\varDelta_{ij} = 0$ if $d(i,j) > 1$, where $d(i,j)$ is the Euclidean distance between the sites i and j, and $\overline{z}_i = 3$, $\underline{z}_i = 0$ for all i.

The matrix $\varDelta$ is the collection of the toppling rules, and is conveniently seen as a set of row vectors $\boldsymbol{\Delta}_i = \{\varDelta_{ij}\}_{1 \leq j \leq n}$. Denote by t_i the action of a toppling at i. If such a toppling occurs, the configuration $\mathbf{z}$ is transformed according to

$$t_i \mathbf{z} = \mathbf{z} - \boldsymbol{\Delta}_i \,. \tag{6}$$

A site i is *positively-unstable* (or just *unstable*) if $z_i > \overline{z}_i$. In this case, and only in this case, a toppling can be performed at i. Note that, after the toppling, it is still $z_i \geq \underline{z}_i$ (more precisely, $z_i > \overline{z}_i - \varDelta_{ii}$), while z_j's for $j \neq i$ have not decreased, thus the topplings leave stable both spaces S_+ and S'_+. The *relaxation operator* $\mathcal{R}$ is the map from S_+ to S, coinciding with the identity on S, that associates to a configuration $\mathbf{z} \in S_+$ the unique configuration $\mathcal{R}(\mathbf{z}) \in S$ resulting from the application of topplings at unstable sites. Unicity relies on the abelianity of the toppling rules, i.e. that, if i and j are both unstable for $\mathbf{z}$, j is unstable for $t_i\mathbf{z}$, and thus crucially relies on the fact that $\varDelta_{ij} \leq 0$ for $i \neq j$.

A site i is *negatively-unstable* if $z_i < \underline{z}_i$. In this case, an *antitoppling* can be performed at i, with the rule

$$t_i^\dagger \mathbf{z} = \mathbf{z} + \boldsymbol{\Delta}_i \,. \tag{7}$$

We use deliberately the symbol $t_i^\dagger$ instead of t_i^{-1} because, although the effect of the linear transformation (7) is just the inverse of the effect of (6), the *if* conditions for applicability of the two operators are different.

Now antitopplings leave stable both spaces S_- and S'_-. The *antirelaxation operator* $\mathcal{R}^\dagger$ is the map from S_- to S, coinciding with the identity on S, that associates to a configuration $\mathbf{z} \in S_-$ the unique configuration $\mathcal{R}^\dagger(\mathbf{z}) \in S$ resulting from the application of antitopplings at negatively-unstable sites. As a matter of fact, the involution

$$\iota \;:\quad \mathbf{z} \to \overline{\mathbf{z}} + \underline{\mathbf{z}} - \mathbf{z} \tag{8}$$

exchanges the role of operators with and without the † suffix, i.e., for the operators above and all the others introduced later on, we have $A^\dagger(\mathbf{z}) \equiv \iota A(\iota \mathbf{z})$.

Note that, for a configuration $\mathbf{z} \in \mathbb{Z}^n$, we cannot exchange in general the ordering of topplings and antitopplings (i.e., if i and j are respectively positively- and negatively-unstable for $\mathbf{z}$, j might be stable for $t_i\mathbf{z}$, and i might be stable for $t_j^\dagger\mathbf{z}$). Consistently, we do *not* define any relaxation-like operator from $\mathbb{Z}^n$ to S, as it would be intrinsically ambiguous, and even not finite, even when the toppling matrix is dissipative [2].

Remark, however, that the definition of $\mathcal{R}$ can be trivially extended in order to map unambiguously $\mathbb{Z}^n$ to S_-, by letting it produce a toppling only on unstable

[2] For example, in a sandpile that has all $\overline{z} = 2$, all $\underline{z} = 0$, all diagonal $\varDelta_{aa} = 3$, and non-zero $\varDelta_{ab}$ with $a = i, j, k$ given by the list $\{\varDelta_{ij}, \varDelta_{jk}, \varDelta_{ki}, \varDelta_{ii_{1,2}}, \varDelta_{jj_{1,2}}, \varDelta_{kk_{1,2}}\}$ and all equal to -1, in the configuration $\mathbf{z}$ depicted below one can repeat infinitely many times the

sites, and, analogously, $\mathcal{R}^\dagger$ to $\mathbb{Z}^n$ to S_+ (and both $\mathcal{R}^\dagger\mathcal{R}$ and $\mathcal{R}\mathcal{R}^\dagger$ map $\mathbb{Z}^n$ to S, but $\mathcal{R}^\dagger(\mathcal{R}(\mathbf{z})) \neq \mathcal{R}(\mathcal{R}^\dagger(\mathbf{z}))$ in general).

On the space $\mathbb{Z}^n$ we can of course take linear combinations, and define the *sum* of two configurations, $\mathbf{z}+\mathbf{w}$, and the *multiplication by scalars* $k \in \mathbb{Z}$, $k\mathbf{z}$. For generic $\overline{\mathbf{z}}$ and $\underline{\mathbf{z}}$ these operations do not leave stable any of the subsets $S_\pm$ and S. But the sum and difference, $\mathbf{z}+\mathbf{w}$ and $\mathbf{z}-\mathbf{w}$, are binary operations in the following spaces:

$$\mathbf{z}+\mathbf{w}: \qquad S_+ \times \Omega \to S_+\,; \qquad S'_+ \times \Omega \to S'_+\,; \tag{9}$$

$$\mathbf{z}-\mathbf{w}: \qquad S_- \times \Omega \to S_-\,; \qquad S'_- \times \Omega \to S'_-\,. \tag{10}$$

Thus, in particular, the two maps $\mathcal{R}(\mathbf{z}+\mathbf{w})$ and $\mathcal{R}^\dagger(\mathbf{z}-\mathbf{w})$ can act from $S \times \Omega$ into S. This suggests to give a special name and symbol to the simplest family of these binary operations, seen as operators on S. Call $\mathbf{e}_i$ the canonical basis of $\mathbb{Z}^n$, i.e. $(\mathbf{e}_i)_j = \delta_{i,j}$. Define the operators $\hat{a}_i$ such that $\hat{a}_i\mathbf{z} = \mathbf{z}+\mathbf{e}_i$, and introduce the operators of *sand addition* and *removal*

$$a_i = \mathcal{R}\hat{a}_i\,; \qquad a_i^\dagger = \mathcal{R}^\dagger\hat{a}_i^{-1}\,. \tag{11}$$

The a_i's commute among themselves, i.e., for every $\mathbf{z}$, $a_i a_j \mathbf{z} = a_j a_i \mathbf{z} = \mathcal{R}(\mathbf{z}+\mathbf{e}_i+\mathbf{e}_j)$. Similarly, the $a_i^\dagger$'s commute among themselves. More generally, for any $\mathbf{z} \in \mathbb{Z}^n$ and $\mathbf{w} \in \Omega$, $\mathcal{R}(\mathcal{R}(\mathbf{z})+\mathbf{w}) = \mathcal{R}(\mathbf{z}+\mathbf{w})$, and $\mathcal{R}^\dagger(\mathcal{R}^\dagger(\mathbf{z})-\mathbf{w}) = \mathcal{R}^\dagger(\mathbf{z}-\mathbf{w})$, this implying

$$\mathcal{R}(\mathbf{z}+\mathbf{w}) = \Big(\prod_{i=1}^{n}(a_i)^{w_i}\Big)\,\mathbf{z}\,; \qquad \mathcal{R}^\dagger(\mathbf{z}-\mathbf{w}) = \Big(\prod_{i=1}^{n}(a_i^\dagger)^{w_i}\Big)\,\mathbf{z}\,. \tag{12}$$

This can be seen by induction in i, as $\mathcal{R}(\mathbf{z}+(\mathbf{w}'+\mathbf{e}_i)) = \mathcal{R}(\mathcal{R}(\mathbf{z}+\mathbf{e}_i)+\mathbf{w}') = \mathcal{R}((a_i\mathbf{z})+\mathbf{w}'))$. For later convenience, for $\mathbf{w} \in \Omega$, we introduce the shortcuts

$$a_{\mathbf{w}} = \prod_{i=1}^{n}(a_i)^{w_i} = \mathcal{R}(\,\cdot\,+\mathbf{w})\,; \qquad a_{\mathbf{w}}^\dagger = \prod_{i=1}^{n}(a_i^\dagger)^{w_i} = \mathcal{R}^\dagger(\,\cdot\,-\mathbf{w})\,; \tag{13}$$

(note that the order in the products does not matter).

These properties have an important consequence on the structure of the Markov Chain dynamics introduced in Section 1: at each integer time t a site $i(t)$ is chosen at random, and $\mathbf{z}(t+1) = a_{i(t)}\mathbf{z}(t)$. As long as we are interested in the configuration $\mathbf{z}(t_{\mathrm{fin}})$ for a unique final time t_{fin}, for a given initial state $\mathbf{z}(0)$, it is not necessary to follow the entire evolution $\mathbf{z}(t)$, for $0 \le t \le t_{\mathrm{fin}}$, but it is enough to take the vector $\mathbf{w} = \sum_t \mathbf{e}_{i(t)}$, and evaluate $\mathcal{R}(\mathbf{z}(0)+\mathbf{w})$. Thus, the final result of the time evolution depends on the set $\{i(t)\}_{0\le t<t_{\max}}$ of moves at all times, in a way which is invariant under their permutations.

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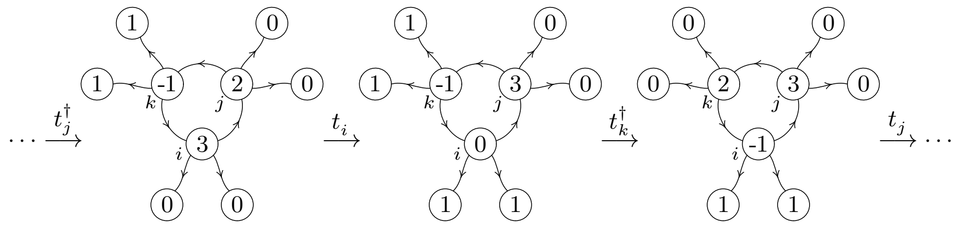
cycle $(t_j^\dagger t_k t_i^\dagger t_j t_k^\dagger t_i)\mathbf{z} = \mathbf{z}$.



Note however that, similarly to topplings with antitopplings, also each operator of sand addition and removal do not commute, i.e. $a_i a_j^\dagger \mathbf{z} \neq a_j^\dagger a_i \mathbf{z}$ in general, even for $\mathbf{z} \in S$. Therefore, in a Markov process involving both a_i's and $a_i^\dagger$'s, in order to know the final configuration, it is necessary to follow the full trace of the time evolution. We will describe and briefly investigate a collection of dynamics of this kind, for the sandpile on a square lattice, in Section 4. Furthermore, see [31] for a first extensive investigation of a dynamics in this family.

Beside the commutativity relations $a_i a_j = a_j a_i$ (and conjugated ones), there is a collection of n relations, encoded by the toppling matrix: for any i, when acting on configurations such that $z_i > \overline{z}_i - \Delta_{ii}$, (respectively, such that $z_i < \underline{z}_i + \Delta_{ii}$), we have

$$a_i^{\Delta_{ii}} = \prod_{j \neq i} a_j^{-\Delta_{ij}} \, ; \qquad\qquad (a_i^\dagger)^{\Delta_{ii}} = \prod_{j \neq i} (a_j^\dagger)^{-\Delta_{ij}} \, . \tag{14}$$

This because $t_i \hat{a}_i^{\Delta_{ii}} = \prod_{j \neq i} \hat{a}_j^{-\Delta_{ij}}$ on such configurations, and the site i is certainly positively-unstable after the application of $\hat{a}_i^{\Delta_{ii}}$. As a corollary, relations valid for generic configurations in S_+, and in S_-, respectively, are

$$a_i^{\overline{z}_i - \underline{z}_i + 1} = a_i^{\overline{z}_i - \underline{z}_i + 1 - \Delta_{ii}} \prod_{j \neq i} a_j^{-\Delta_{ij}} \, ; \tag{15}$$

$$(a_i^\dagger)^{\overline{z}_i - \underline{z}_i + 1} = (a_i^\dagger)^{\overline{z}_i - \underline{z}_i + 1 - \Delta_{ii}} \prod_{j \neq i} (a_j^\dagger)^{-\Delta_{ij}} \, ; \tag{16}$$

which are concisely written as

$$a_i^{\overline{z}_i - \underline{z}_i + 1} = a_{-\boldsymbol{\Delta}_i + (\overline{z}_i - \underline{z}_i + 1)\mathbf{e}_i} \, ; \tag{17}$$

$$(a_i^\dagger)^{\overline{z}_i - \underline{z}_i + 1} = a^\dagger_{-\boldsymbol{\Delta}_i + (\overline{z}_i - \underline{z}_i + 1)\mathbf{e}_i} \, . \tag{18}$$

Remark that the two sets of equations (14) and (15) do coincide if the sandpile is tight, i.e. if $\overline{z}_i - \underline{z}_i + 1 = \Delta_{ii}$ for all i.

In the previous paragraphs we only considered sums $\mathbf{z} + \mathbf{w}$, in which one of the two configurations is taken from a space among S, $S_\pm$, or $S'_\pm$, and the other one from Ω, or other spaces with no dependence from $\overline{\mathbf{z}}$ and $\underline{\mathbf{z}}$.

In such a situation, we have a clear covariance of the notations under an overall translation of the coordinates. I.e., for any $\mathbf{r} \in \mathbb{Z}^n$, under the map $\mathbf{z} \to \mathbf{z} + \mathbf{r}$, we have an isomorphism between the sandpile $\mathcal{A}(\Delta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$ and $\mathcal{A}(\Delta, \overline{\mathbf{z}} + \mathbf{r}, \underline{\mathbf{z}} + \mathbf{r})$. We call a *gauge invariance* of the model the covariance explicitated above, and a *gauge fixing* any special choice of offset vector $\mathbf{r}$.

As always, a gauge fixing reduces the apparent number of parameters in the model. Some special choices simplify the notations in certain contexts. For example, we can set $\underline{\mathbf{z}} = \mathbf{0}$, so that $S_+ \equiv \Omega$, that we call the $\underline{\mathbf{z}} = \mathbf{0}$ *gauge*, or $\overline{z}_i = \Delta_{ii} - 1$, so that $S'_+ \equiv \Omega$, that we call the *positive-cone gauge*.

In particular, the positive-cone gauge is the most natural one in the Abelian Sandpile in which antitoppling are not considered, as in this case the parameters $\underline{\mathbf{z}}$ do not play any role. Conversely, the covariant formalism is the only formulation that does not break explicitly the involution symmetry implied by (8).

2.2 Statement of results

Two new theoretical results are extensively required for the analysis of interesting dynamics, in Section 4. The first one is the following

Theorem 1 *For every $i \in [n]$, acting on S_+,*

$$a_i a_i^\dagger a_i = a_i\,; \tag{19}$$

and, acting on S_-,

$$a_i^\dagger a_i a_i^\dagger = a_i^\dagger\,. \tag{20}$$

This theorem will be proven in the next subsection. As an aside, we know that Theorem 1 is the tip of the iceberg of a much more general family of identities, with operators a_i and $a_i^\dagger$ replaced by $a_{\mathbf{w}}$ and $a_{\mathbf{w}}^\dagger$, for $\mathbf{w} \in \Omega$, that we plan to elucidate in forthcoming works. Note for example how, acting on S_+, the identity $a_{\mathbf{w}} a_{\mathbf{w}}^\dagger a_{\mathbf{w}} = a_{\mathbf{w}}$ holds for simple reasons if $\mathbf{w}$ is *recurrent*, due to the fact that $a_{\mathbf{w}} a_{\mathbf{w}}^\dagger a_{\mathbf{w}} \mathbf{z} \sim a_{\mathbf{w}} \mathbf{z}$, both sides of the equation are stable recurrent, and there exists a unique stable recurrent representative in each equivalence class (see later Section 2.3 for the pertinent definitions).

An immediate corollary of Theorem 1 is the following

Corollary 1 *For every $i \in [n]$, acting on S, $a_i^\dagger a_i$ and $a_i a_i^\dagger$ are idempotents, i.e., $(a_i^\dagger a_i)^2 = a_i^\dagger a_i$, and $(a_i a_i^\dagger)^2 = a_i a_i^\dagger$.*

Indeed, the set $S = S_+ \cap S_-$ is left stable by the action of the monoid. It is enough to multiply equation (19) by $a_i^\dagger$, from the left or from the right respectively for the two claims, or, alternatively, multiply (20) by a_i, from the right or from the left respectively.

The simplicity of Corollary 1 may suggest that abelianity is restored at the level of these idempotent combinations. This is not the case. No pairs of distinct operators in the set $\{a_1^\dagger a_1, \dots, a_n^\dagger a_n, a_1 a_1^\dagger, \dots, a_n a_n^\dagger\}$ commute with each other, in general. Nonetheless, a few interesting facts are found.

For a finite set $I \subseteq [n]$, call $\mathcal{N}_I = \{a_i^\dagger a_i\}_{i \in I}$. For $X \subseteq \mathbb{Z}^n$, call $\mathcal{N}_I[X]$ the set of $\mathbf{y} \in \mathbb{Z}^n$ such there exists a configuration $\mathbf{x} \in X$ and a finite sequence $(i_1, \dots, i_k)$ of elements in I such that $a_{i_k}^\dagger a_{i_k} a_{i_{k-1}}^\dagger a_{i_{k-1}} \cdots a_{i_1}^\dagger a_{i_1} \mathbf{x} = \mathbf{y}$, that is $\mathcal{N}_I[X]$ is the set of possible images of X under the action of products of operators in $\mathcal{N}_I$.

The second new theoretical result of this paper is

Theorem 2 *Consider a sandpile $\mathcal{A}(\Delta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$ such that $\Delta = \Delta^{\mathrm{T}}$ and $\Delta_{ij} \in \{0, -1\}$ for $i \neq j$. For any $I \subseteq [n]$, and any $\mathbf{z} \in S_+$, there exists a unique state $\mathbf{y}(\mathbf{z}, I)$ in $\mathcal{N}_I[\{\mathbf{z}\}]$ such that $a_i^\dagger a_i \mathbf{y} = \mathbf{y}$ for all $i \in I$. For any state $\mathbf{x} \in \mathcal{N}_I[\{\mathbf{z}\}]$, we also have $\mathbf{y} \in \mathcal{N}_I[\{\mathbf{x}\}]$.*

Thus, this theorem shows that certain collections of idempotents have a well-characterised set of common fixed points. The portion of this set accessible from any configuration $\mathbf{z}$ has cardinality exactly 1.

The statement of this theorem can be translated in terms of a stochastic evolution. Consider a Markov Chain in which the initial state is $\mathbf{z}(0) = \mathbf{z}$, and at each time t an element $a_{i_t}^\dagger a_{i_t} \in \mathcal{N}_I$ is chosen (with non-zero probabilities for all elements), and $\mathbf{z}(t+1) = a_{i_t}^\dagger a_{i_t} \mathbf{z}(t)$. Then the theorem states that this Markov Chain is absorbent, on an unique state $\mathbf{y}$, and in particular, no matter the evolution up to some time t, $\mathbf{y}$ is still accessible from $\mathbf{z}(t)$ (and in fact it *will* be reached at some time). A dynamics of this kind will be described in Section 4.3.

The theorem above will be proven in Section 3. Furthermore, the state $\mathbf{y}(\mathbf{z}, I)$ will be shown to have a further characterization, in terms of a multitoppling Abelian Sandpile associated to the original system.

2.3 Further aspects of the theory

There exists a natural equivalence relation on vectors $\mathbf{z} \in \mathbb{Z}^n$, that partitions this set into $\det \Delta$ classes, which are affine subspaces of $\mathbb{Z}^n$, all isomorphic under translation. This notion was first introduced and studied in [35]. We recall here only briefly the easiest facts.

We say that $\mathbf{z} \sim \mathbf{w}$ if there exists $\mathbf{T} \in \mathbb{Z}^n$ such that $\mathbf{z} - \mathbf{w} = \mathbf{T}\Delta$. In particular, as $\mathbf{z} - t_i\mathbf{z} = -(\mathbf{z} - t_i^\dagger \mathbf{z}) = \boldsymbol{\Delta}_i = \mathbf{e}_i\Delta$, we have $\mathbf{z} \sim \mathcal{R}(\mathbf{z}) \sim \mathcal{R}^\dagger(\mathbf{z})$. Analogously, as $\hat{a}_i^{\Delta_{ii}}\mathbf{z} - \left(\prod_{j\neq i} \hat{a}_j^{-\Delta_{ij}}\right)\mathbf{z} = -\mathbf{e}_i\Delta$, we get $a_i^{\Delta_{ii}}\mathbf{z} \sim \left(\prod_{j\neq i} a_j^{-\Delta_{ij}}\right)\mathbf{z}$ for all i, as it should at the light of (14). The dissipativity condition on the toppling matrix ensure that $\det \Delta \neq 0$. Thus Δ^{-1} exists, and it is evident that $\mathbf{z} \sim \mathbf{w}$ if and only if $\mathbf{z}\Delta^{-1} - \mathbf{w}\Delta^{-1} \in \mathbb{Z}^n$. So, the fractional parts $Q_i^{(\mathrm{frac})}(\mathbf{z}) = (\mathbf{z}\Delta^{-1})_i - \lfloor(\mathbf{z}\Delta^{-1})_i\rfloor$, called the *charges* of the configuration $\mathbf{z}$, completely identify the equivalence class of $\mathbf{z}$. As a corollary, $Q_i^{(\mathrm{frac})}(\mathbf{z}) = Q_i^{(\mathrm{frac})}(t_i\mathbf{z}) = Q_i^{(\mathrm{frac})}(t_i^\dagger\mathbf{z})$ (when t_i or $t_i^\dagger$ are applicable to $\mathbf{z}$). For future convenience, we also define $\mathbf{Q}(\mathbf{z}) = (\mathbf{z}\Delta^{-1})$, so that $Q_i^{(\mathrm{frac})} = Q_i - \lfloor Q_i \rfloor$ [3].

The set S of stable configurations is divided into the two subsets of *stable transient* and *stable recurrent* configurations, $S = T \cup R$. Several equivalent characterizations of recurrency exist, some of which extend naturally to S_+, and even to the full space $\mathbb{Z}^n$ (this is also our choice).

In particular, we give three definitions, all valid in $\mathbb{Z}^n$. Under all definitions, a configuration is *transient* if it is not recurrent.

Definition 1 *A configuration* $\mathbf{z}$ *is* recurrent by identity test *if there exists a permutation* $\sigma \in \mathfrak{S}_n$ *such that* $t_{\sigma(n)} \cdots t_{\sigma(2)} t_{\sigma(1)}(\mathbf{z} + \mathbf{b}^+) = \mathbf{z}$ *is a valid toppling sequence.*

Definition 2 *A configuration* $\mathbf{z}$ *is* recurrent by toppling covering *if there exists a configuration* $\mathbf{u}$*, such that* $\mathbf{z} = t_{i_k} \cdots t_{i_1}\mathbf{u}$ *is a valid toppling sequence, and at least one toppling is performed at each site.*

Definition 3 *A configuration* $\mathbf{z}$ *is* recurrent by absence of FSC's *if, for every set* I *of sites, there exists* $i \in I$ *such that* $z_i > \overline{z}_i - \sum_{j\in I} \Delta_{ji}$.

The reasonings of the following paragraphs will prove, among other things, that these three definitions are equivalent.

First of all, a configuration recurrent by identity test is also recurrent by toppling covering (one can take $\mathbf{u} = \mathbf{z} + \mathbf{b}^+$).

Note that, if $\mathbf{u} = \mathcal{R}(\mathbf{v})$ and $\mathbf{v}$ is recurrent by toppling covering, also $\mathbf{u}$ is recurrent by toppling covering. Furthermore, as after a toppling t_i one has $z_i > \overline{z}_i - \Delta_{ii}$, all the

[3] This topic would deserve some more lines. The charges $Q_i^{(\mathrm{frac})}(\mathbf{z})$ have two disadvantages: they are redundant in general, and require “fractional parts”, instead of integer arithmetics. Both these issues are solved through the identification of a different set of charges, $I_i(\mathbf{z})$. A result of the classical theory of the Smith Normal Form is that any non-singular $n \times n$ matrix Δ can be decomposed as $\Delta = ADB$, where A and B are square matrices with determinant ± 1, and $D = \mathrm{diag}(d_i)$, with $d_1 \geq d_2 \geq \cdots d_n$, all matrices having integer coefficients. The decomposition is not unique, but the d_i's are, and have a further characterization that makes unicity manifest. Call g_i the greatest common divisor among the minors of Δ of size $n - i$, and pose $g_n = 1$. Then $d_i = g_{i-1}/g_i$. One defines $I_i^{(0)}(\mathbf{z}) = (A^{-1}\mathbf{z})_i$, and $0 \leq I_i < d_i$, with $I_i \equiv I_i^{(0)}$ modulo d_i. Dropping the trivial charges, associated to $d_i = 1$, the new set of charges is non-redundant and completely identifies the equivalence classes. The new charges are related to the previous ones through $\mathbf{I} = DB\,\mathbf{Q}$. A more detailed treatment can be found in the original reference, [35]. We do not make use of these improved charges in the present paper.

recurrent configurations, either by toppling covering or by identity test, are in fact contained in S'_+. This last reasoning is a first example of *forbidden sub-configuration* (FSC), whose generalisation involves more than one site at a time. For $\mathbf{z} \in \mathbb{Z}^n$, and $I \subseteq [n]$, define $\mathbf{z}|_I$ as the restriction to the components z_i with index $i \in I$.

For a set I, define the vector $\mathbf{f}_{\max}(I) \in \mathbb{Z}^I$ as

$$\big(f_{\max}(I)\big)_i = \overline{z}_i - \sum_{j \in I} \varDelta_{ji} \,. \tag{21}$$

We say that the pair $(I, \mathbf{z}|_I)$ is a *forbidden sub-configuration* for $\mathbf{z}$ if $\mathbf{z} \preceq \mathbf{f}_{\max}(I)$. It is straightforward to recognize that $\mathbf{z}$ is recurrent for absence of FSC's if and only if, for all I, $\mathbf{z}|_I \not\preceq \mathbf{f}_{\max}(I)$, thus legitimating the terminology.

A collection of the pairs $(I, \mathbf{f}_{\max}(I))$ with smallest $|I|$ in the BTW sandpile is as follows:

$$\begin{array}{|c|}\hline -1\\\hline\end{array}\qquad \begin{array}{|c|c|}\hline 0&0\\\hline\end{array}\qquad \begin{array}{|c|}\hline 0\\\hline 0\\\hline\end{array}\qquad \begin{array}{|c|c|c|}\hline 0&1&0\\\hline\end{array}\qquad \begin{array}{|c|c|}\cline{1-1} \multicolumn{1}{|c|}{0} & \multicolumn{1}{c}{} \\\hline 1&0\\\hline\end{array}\quad \dots\quad \begin{array}{|c|c|c|c|}\hline 0&1&1&0\\\hline\end{array}\quad \dots\quad \begin{array}{|c|c|}\hline 1&1\\\hline 1&1\\\hline\end{array}\quad \dots$$

The connection between the definitions of recurrent by toppling covering and by absence of FSC's is given by the following statement, slightly more general that what would suffice at this purpose.

Proposition 1 *Let $\mathbf{u} \in S'_+$ and $\mathbf{v} = t_{i_k} t_{i_{k-1}} \cdots t_{i_1} \mathbf{u}$. Define $A = \bigcup_{1 \leq a \leq k} \{i_a\}$. Then, for all B such that $|B \smallsetminus A| \leq 1$, $\mathbf{v}|_B \not\preceq \mathbf{f}_{\max}(B)$.*

This proposition implies as a corollary (for $|B \smallsetminus A| = 0$) that configurations which are recurrent by toppling covering are also recurrent by absence of FSC's. The case $|B \smallsetminus A| = 1$ also emerges naturally from the proof.

Proof. The claim $\mathbf{v}|_B \not\preceq \mathbf{f}_{\max}(B)$ can be restated as the existence of $s \in B$ such that $v_s > f_{\max}(B)_s$. We will produce a valid choice for s.

If $|B \smallsetminus A| = 1$, choose as s the only site in B and not in A. Let $\mathbf{u}' = \mathbf{u}$ and $\tau(s) = 0$ in this case. Otherwise, for all $i \in B$, call $\tau(i)$ the maximum $1 \leq a \leq k$ such that $i_a = i$, then choose as s the index realising the minimum of $\tau(i)$, i.e., the site of B that has performed its last toppling more far in the past. Call $\mathbf{u}' = t_{i_{\tau(s)}} \cdots t_{i_1} \mathbf{u}$, the configuration obtained after the last toppling in s.

Note that, as $\mathbf{u} \in S'_+$ and this space is stable under topplings, $u'_s \geq \overline{z}_s - \varDelta_{ss} + 1$. In the remaining part of the avalanche, no more topplings occur at s. Furthermore, all the other sites $j \in B$ do topple at least once, at $a = \tau(j)$. Thus we have

$$\begin{aligned} v_s &= u'_s - \sum_{a=\tau(s)+1}^{k} \varDelta_{i_a s} \geq \overline{z}_s - \varDelta_{ss} + 1 - \sum_{a=\tau(s)+1}^{k} \varDelta_{i_a s} \\ &= \overline{z}_s - \sum_{i \in B} \varDelta_{is} + 1 - \sum_{\substack{\tau(s) < a \leq k \\ a \notin \{\tau(j)\}_{j \in B}}} \varDelta_{i_a s} \geq \overline{z}_s - \sum_{i \in B} \varDelta_{is} + 1 \,. \end{aligned} \tag{22}$$

The comparison with the definition (21) of $\mathbf{f}_{\max}(B)$ allows to conclude. □

We have now the ingredients to prove Theorem 1.

Proof of Theorem 1. The two equations are related by the involution, so we only prove (19). I.e., for all $\mathbf{z} \in S_+$, $a_i a_i^\dagger a_i \mathbf{z} = a_i \mathbf{z}$. First of all, as, for $\mathbf{z} \in S_+$ and $\mathbf{w} \in \varOmega$, $a_{\mathbf{w}} \mathbf{z} = a_{\mathbf{w}} \mathcal{R}(\mathbf{z})$, we can restrict our attention to $\mathbf{z} \in S$. If $z_i < \overline{z}_i$ we have $a_i^\dagger a_i \mathbf{z} = \mathbf{z}$,

and our relation follows. If $z_i = \overline{z}_i$, the avalanche due to the action of a_i performs at least one toppling at i. By Proposition 1, $\mathbf{y} = a_i\mathbf{z}$ has no FSC's with $I = \{i\}$ or $I = \{i,j\}$. That is, either $y_i > \overline{z}_i - \Delta_{ii}$, or $y_i = \overline{z}_i - \Delta_{ii}$ and, for all $j \neq i$, $y_j \geq \overline{z}_j - \Delta_{jj} - \Delta_{ij}$. By direct inspection, in the first case $a_i a_i^\dagger \mathbf{y} = \hat{a}_i\hat{a}_i^{-1}\mathbf{y} = \mathbf{y}$ and in the second case $a_i a_i^\dagger \mathbf{y} = t_i \hat{a}_i t_i^\dagger \hat{a}_i^{-1}\mathbf{y} = \mathbf{y}$. In both cases, our relation follows. □

For a sandpile $\mathcal{A}(\Delta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$, call $\mathcal{A}|_I$ the sandpile described by the toppling matrix $\Delta_{I,I}$ the principal minor of Δ corresponding to the rows and columns in I, and by the threshold vectors $\overline{\mathbf{z}}|_I$ and $\underline{\mathbf{z}}|_I$. Also call $\mathbf{b}^+(I)$ the vector $\mathbf{b}^+$ associated to $\Delta_{I,I}$. The observation that ultimately allows to relate the definitions of recurrency by absence of FSC's and by identity test is the fact that

$$\mathbf{f}_{\max}(I) + \mathbf{b}^+(I) = \overline{\mathbf{z}}|_I\,. \tag{23}$$

A first remark in this direction is that, if $I = [n]$ is not a FSC for $\mathbf{z}$, we can at least start the avalanche in the definition of recurrent by identity test, i.e. we have at least one site $i = \sigma(1)$ which is unstable. Indeed, we have at least one i such that $z_i > f_{\max}([n])_i$. Then, by (23), $(z + b^+)_i > f_{\max}([n])_i + b_i^+ = \overline{z}_i$.

As (23) holds for any set I, the reasoning above works for any restricted sandpile $\mathcal{A}|_I$. Suppose to have a configuration $\mathbf{z}^{(I)}$ which is recurrent in $\mathcal{A}|_I$ by absence of FSC's. Then we have at least one site $i \in I$ which is unstable, because we have at least one i such that $z_i^{(I)} > f_{\max}(I)_i$, and $(z^{(I)} + b^+(I))_i > f_{\max}(I)_i + b^+(I)_i = \overline{z}_i$.

This allows to construct an induction. Let $I' = I \setminus i$, $\mathbf{v}^{(I')} = (t_i\mathbf{z}^{(I)})|_{I'}$ and $\mathbf{z}^{(I')} = \mathcal{R}(\mathbf{v}^{(I')})$. For $j \in I'$, $v_j^{(I')} = z_j^{(I)} - \Delta_{ij} \geq z_j^{(I)}$. Remark that the definition of $\mathbf{f}_{\max}(J)$ is the same on any restricted sandpile $\mathcal{A}_I$ with $I \supseteq J$. Thus, as $\mathbf{z}^{(I)}$ is recurrent for absence of FSC's, and this property is preserved under relaxation (by Proposition 1), also $\mathbf{z}^{(I')}$ is recurrent by absence of FSC's, on $\mathcal{A}|_{I'}$. This, together with equation (23), gives the induction step.

In summary, we can perform the complete avalanche $t_{\sigma(n)} \cdots t_{\sigma(1)}$, algorithmically, by initialising $\mathbf{z}^{(0)} = \mathbf{z} + \mathbf{b}^+$ and $I_0 = [n]$, and, for $a = 0, \ldots, n-1$, i_a is any site i at which $\mathbf{z}^{(a)}$ is unstable (which is proven to exist by the reasoning above), $I_{a+1} = I_a \setminus i_a$, and $\mathbf{z}^{(a+1)} = \mathcal{R}(t_{i_a}\mathbf{z}^{(a)})$.

This completes the equivalence of our three definitions of recurrent configurations, thus from now on we will omit to specify the defining property.

A directed graph can be associated to a sandpile, such that $-\Delta_{ij}$ directed edges connect i to j. This completely encodes the off-diagonal part of Δ. The remaining parameters, in particular $\mathbf{b}^+$ and $\mathbf{b}^-$, can be encoded through directed edges incoming from, our outgoing to, a special *sink* vertex. In the undirected case, $\Delta = \Delta^{\mathrm{T}}$, a bijection, called *Burning Test*, relates stable recurrent configurations to spanning trees [7]. A generalisation, called *Script Algorithm* [36], extends this result also to directed graphs even in presence of greedy sites. Through Kirchhoff Matrix-Tree Theorem, one thus gets in this case that the number of stable recurrent configurations is $\det \Delta$.

The configuration $\mathbf{p} = \overline{\mathbf{z}}+\mathbf{1}-\mathcal{R}(\overline{\mathbf{z}}+\mathbf{1})$ has two interesting properties: $p_i \geq 1$ for all i, and $\mathbf{p} \sim \mathbf{0}$. This implies that, for every $\mathbf{z} \in \mathbb{Z}^n$, the iteration of the map $\mathbf{z} \to \mathcal{R}(\mathbf{z}+\mathbf{p})$ must reach a fixed point, in R, the subset of S containing recurrent configurations (this would be true also with $\mathbf{b}^+$ instead of $\mathbf{p}$, but slightly harder to prove). Indeed, calling $c = \max_i(\overline{z}_i - z_i + 1)$, as $\mathcal{R}(\mathcal{R}(\cdots\mathcal{R}(\mathbf{z}+\mathbf{p})\cdots+\mathbf{p})+\mathbf{p}) = \mathcal{R}(\mathbf{z}+c\,\mathbf{p})$, and $\mathbf{z}+c\,\mathbf{p}$ is unstable at all sites, $\mathcal{R}(\mathbf{z}+c\,\mathbf{p})$ is both stable and recurrent (by toppling covering), so it must be in R.

This reasoning proves that each equivalence class has at least one representative in R. In the case $\mathbf{b}^+ \in \Omega$, as both the cardinality of R and the number of classes are

$\det \Delta$, each equivalence class must have a unique representative in R. In particular, the representative in R of $\mathbf{0}$ is called the *recurrent identity*, and we denote it with the symbol Id_r. This configuration can be found as the fixed point of the map $\mathbf{z} \to \mathcal{R}(\mathbf{z} + \mathbf{b}^+)$, started from $\mathbf{0}$ [37,38], or of the map $\mathbf{z} \to \mathcal{R}(\mathbf{z} + \mathbf{p})$, or, under the mild assumption that $\overline{\mathbf{z}} \in \Omega$, more directly, with no need of iterations, by the relation [39]

$$\mathrm{Id}_r = \mathcal{R}(\overline{\mathbf{z}} + (\overline{\mathbf{z}} - \mathcal{R}(\overline{\mathbf{z}} + \overline{\mathbf{z}}))) \,. \tag{24}$$

Indeed, as $\overline{\mathbf{z}} - \mathcal{R}(\overline{\mathbf{z}} + \overline{\mathbf{z}}) \in \Omega$, $\overline{\mathbf{z}} + (\overline{\mathbf{z}} - \mathcal{R}(\overline{\mathbf{z}} + \overline{\mathbf{z}}))$ is recurrent, thus its relaxation is in R.

In the $\underline{\mathbf{z}} = \mathbf{0}$ gauge, the operation $\mathbf{u} \oplus \mathbf{v} := \mathcal{R}(\mathbf{u} + \mathbf{v})$ sends $S \times S$ into S, and thus defines a semigroup on this space. Furthermore, this operation also sends $S \times R \to R$, $R \times S \to R$, and, as a corollary, $R \times R \to R$.

The charges behave linearly under this operation: $\mathbf{Q}(\mathbf{u} \oplus \mathbf{v}) - \mathbf{Q}(\mathbf{u}) - \mathbf{Q}(\mathbf{v}) \in \mathbb{Z}^n$. The unicity of representatives in R of the equivalence classes allows then to construct inverses, for the action on this space, and thus to promote $\oplus$ to a group action on R, of which Id_r is the group identity. This structure was first introduced by Creutz [37,38], and then investigated in several papers [20,35,39,40].

The covariance of the notations allows to define the operation $\oplus$ in general. We have

$$\mathbf{u} \oplus \mathbf{v} := \mathcal{R}(\mathbf{u} + \mathbf{v} - \underline{\mathbf{z}}) \,; \qquad \qquad \mathbf{u} \oplus^\dagger \mathbf{v} := \mathcal{R}^\dagger(\mathbf{u} + \mathbf{v} - \overline{\mathbf{z}}) \,. \tag{25}$$

Note however that now the charges behave in an affine way, and only the translated versions, $\mathbf{Q}'(\mathbf{u}) = \mathbf{Q}(\mathbf{u}) - \mathbf{Q}(\underline{\mathbf{z}})$ and $\mathbf{Q}''(\mathbf{u}) = \mathbf{Q}(\mathbf{u}) - \mathbf{Q}(\overline{\mathbf{z}})$ respectively for the two operations, have no offset. In particular, the two groups induced by $\oplus$ and $\oplus^\dagger$, as well as the two sets R and $R^\dagger$, are isomorphic but not element-wise coincident, as the only natural bijection among the two makes use of the involution ι.

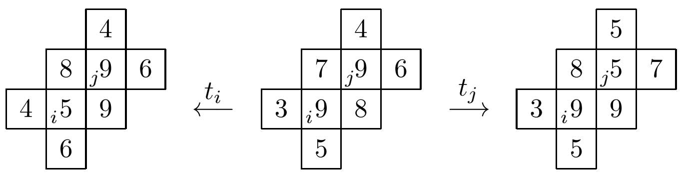
Call $\mathcal{M} = \mathcal{M}[a_i, a_i^\dagger]$ the transition monoid generated by the a_i's and $a_i^\dagger$'s acting on our set of configurations. A generic element in $\mathcal{M}$ has the form

$$A = a^\dagger_{i^1_1} \cdots a^\dagger_{i^1_{\ell(1)}} \, a_{i^2_1} \cdots a_{i^2_{\ell(2)}} \, a^\dagger_{i^3_1} \cdots a^\dagger_{i^3_{\ell(3)}} \, \cdots \, a_{i^{2k}_1} \cdots a_{i^{2k}_{\ell(2k)}} \,. \tag{26}$$

3 Multitopplings in Abelian Sandpiles

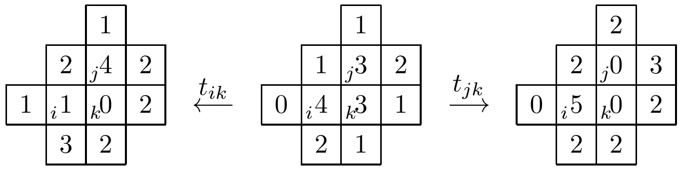
Consider a situation not dissimilar from the BTW sandpile, with height variables z_i at the sites of a square lattice, but let the toppling t_i occur if the *gradient* passes a given threshold. For example, choose to perform a toppling at i if i is a local maximum.

Despite the simplicity of such a rule, abelianity is lost, as is easily verified. For example, on a configuration $\mathbf{z}$ as below, the two sites i and j are unstable, but $t_i\mathbf{z}$ and $t_j\mathbf{z}$ differ, and it can well be that both have no unstable sites.



We describe now another variant of the BTW sandpile in which abelianity is lost. We come back to toppling criteria based on having heights above given thresholds,

as in the ordinary BTW (and not gradients, as above). Now the rule is that, if two neighbouring sites have both height 3 or higher, both their heights are decreased by 3, and the height of each of the 6 neighbours is increased by 1. Again, it is not difficult to imagine situations in which two sites i and j are unstable, but $t_i\mathbf{z}$ and $t_j\mathbf{z}$ differ, and both have no unstable sites, as in the configuration below.



In this section we will describe a family of sandpiles for which abelianity is preserved. In these models we have toppling rules based on "heights above given thresholds", just as in the ordinary ASM and in the example above. In fact, these models generalise the ordinary ASM, because we have simultaneously all the ordinary toppling moves, and a (possibly empty) collection of *multitoppling* moves satisfying certain criteria.

As we will see later on, the moves described in the second example would be a valid choice – and remark indeed how, including also ordinary topplings, the configurations on the left and right of the figure above are still unstable on j and i, respectively, and become identical after these topplings.

Consider an ordinary ASM $\mathcal{A}(\varDelta, \overline{\mathbf{z}}, \underline{\mathbf{z}})$. For simplicity of notations, we set in the positive-cone gauge, so that site i is unstable if $z_i \geq \varDelta_{ii}$.

For a non-empty set $I \subseteq [n]$, call $\boldsymbol{\Delta}_I = \sum_{i \in I} \boldsymbol{\Delta}_i$. A multitoppling operator t_I can be associated to a set I. First of all, $\mathbf{z}$ is unstable for toppling I if $z_j \geq (\varDelta_I)_j$ for all $j \in I$ (note how, in the positive-cone gauge, this coincides with the ordinary definition when $|I| = 1$). Then, if the configuration is unstable, it is legitimate to perform the toppling $t_I\mathbf{z} = \mathbf{z} - \boldsymbol{\Delta}_I$. Also note that the absence of greedy sites implies that, for all $j \in I$, $(\varDelta_I)_j \geq 0$, while for all $j \notin I$, $(\varDelta_I)_j \leq 0$.

Consider a collection $\mathcal{L}$ of non-empty subsets of $[n]$. The interest in multitoppling rules for the Abelian Sandpile Model is in the following fact

Proposition 2 *Suppose that, for every $I, J \in \mathcal{L}$, the sets $I' = I \smallsetminus J$ and $J' = J \smallsetminus I$ are either empty or in $\mathcal{L}$. Then, if $\mathbf{z}$ is unstable for both I and J, $t_I\mathbf{z}$ is unstable for J' and $t_J\mathbf{z}$ is unstable for I'.*

As clearly $t_{J'}t_I\mathbf{z} = t_{I'}t_J\mathbf{z}$, it easily follows

Corollary 2 *In the conditions of Proposition 2, the operator $\mathcal{R}$ is unambiguous.*

Antitopplings $t_I^\dagger$ and antirelaxation $\mathcal{R}^\dagger$ are defined just as in the ordinary case, e.g. through the involution ι, which is still defined as in (8). A configuration is stable if no toppling or antitoppling can occur (this coincides with the definition of S in the ordinary case). Note that, if $\mathcal{L}$ does not contain the atomic set $\{i\}$, S is either empty or of infinite cardinality (because, if $\mathbf{z} \in S$, also $\mathbf{z} + c\,\mathbf{e}_i \in S$ for any $c \in \mathbb{Z}$). In order to exclude this pathological case, we will assume in the following that $\mathcal{L}$ includes the set $\mathcal{L}_0 = \big\{\{i\}\big\}_{i\in[n]}$ of all atomic subsets, i.e. single-site topplings. In this case, as stability w.r.t. $\mathcal{L}$ is a more severe requirement than stability w.r.t. $\mathcal{L}' \subset \mathcal{L}$, we have that S is a subset of the set S_0 of stable configurations in the associated sandpile with only single-site topplings, and thus of finite cardinality. As we will see later on, for any set $\mathcal{L}$ as in Proposition 2 the set S is non-empty, and actually contains a set isomorphic to R, thus it has cardinality bounded below by $|R_0| = \det \varDelta$, an above by $|S_0| = \prod_i (\overline{z}_i - \underline{z}_i + 1)$.

If we require both that $\mathcal{L} \supseteq \mathcal{L}_0$, and satisfies the hypotheses of Proposition 2, we get that $\mathcal{L}$ is a down set in the lattice of subsets, that is, for all $I \in \mathcal{L}$ and $H \subseteq \mathcal{L}$

non-empty, also $H \in \mathcal{L}$ (this is trivially seen: with notations as in Proposition 2, take I and $J = I \smallsetminus H$).

For a multitoppling sandpile $\mathcal{A} = \mathcal{A}(\mathcal{L})$, we will call $\mathcal{A}_0 = \mathcal{A}(\mathcal{L}_0)$ the associated single-site toppling sandpile.

The concept of recurrent configuration has to be reanalysed in this context. The various alternate definitions are modified (in the more complicated situation, but under the simplification of the choice of positive-cone gauge) into

Definition 4 *A configuration* $\mathbf{z}$ *is* recurrent by identity test *if there exists an ordered sequence* $(I_1, \dots, I_k)$ *of subsets of* $[n]$*, constituting a partition of* $[n]$ *(i.e. for all* $i \in [n]$ *there exists a unique* a *such that* $i \in I_a$*), such that* $t_{I_k} \cdots t_{I_2} t_{I_1}(\mathbf{z} + \mathbf{b}^+) = \mathbf{z}$ *is a valid toppling sequence.*

Definition 5 *A configuration* $\mathbf{z}$ *is* recurrent by toppling covering *if there exists a configuration* $\mathbf{u}$*, such that* $\mathbf{z} = t_{I_k} \cdots t_{I_1} \mathbf{u}$ *is a valid toppling sequence, and each site* i *is contained in at least one of the* I_a*'s.*

Definition 6 *A configuration* $\mathbf{z}$ *is* recurrent by absence of FSC's *if, for every set* I *of sites, there exists* $J \in \mathcal{L}$ *with* $L = I \cap J \neq \varnothing$*, and* $z_j \geq \sum_{i \in I \smallsetminus L} \Delta_{ij}$ *for all* $j \in L$*.*

All the reasonings are the immediate generalisation of the ones already done in Section 2.3. We just report here the appropriate modifications in Proposition 1 (recall that in our gauge $\Omega \equiv S'_+$, and is left stable by the topplings).

Proposition 3 *Let* $\mathbf{u} \in \Omega$ *and* $\mathbf{v} = t_{I_k} t_{I_{k-1}} \cdots t_{I_1} \mathbf{u}$*. Define* $A = \bigcup_{1 \leq a \leq k} I_a$*. For any set* B*, there exists a non-empty set* $H \subseteq B$*, such that* $B \subseteq H \cup A$*, and, for all* $s \in H$*,* $v_s > -\sum_{i \in (A \cap B) \smallsetminus H} \Delta_{is}$*.*

PROOF. We will produce explicitly a valid choice of H. If $B \not\supseteq A$, let $H = B \smallsetminus A$ and $\mathbf{u}' = \mathbf{u}$. Note that $(B \cap A) \smallsetminus H = B \cap A$ in this case. If $B \supseteq A$, for all $i \in B$, call $\tau(i)$ the maximum $1 \leq a \leq k$ such that $i \in I_a$, then call $\tau = \max_i \tau(i)$, and $J = I_{\tau(i)}$, i.e., the multitoppling that covered any portion of B more far in the past. Call $\mathbf{u}' = t_{I_\tau} \cdots t_{I_1} \mathbf{u}$, the configuration obtained after this last multitoppling. Note that all the entries of $\mathbf{u}'$ are non-negative. In the remaining part of the avalanche, for some non-empty set $H \subseteq J$, no more topplings occur. Conversely, all the sites $j \in (A \cap B) \smallsetminus H$ do topple at least once (possibly in a multitoppling event). Thus we have, for each $s \in H$,

$$v_s \geq -\sum_{i \in (B \cap A) \smallsetminus H} \Delta_{is}\,, \tag{27}$$

as was to be proven. □

We should modify the concept of forbidden sub-configuration along the same lines. We want to produce pairs $(I, \mathbf{f})$ such that, if $\mathbf{z}$ is recurrent by toppling covering, $\mathbf{z}|_I \neq \mathbf{f}$. For a given I, a vector $\mathbf{f}$ has the property above if, for all $J \in \mathcal{L}$, $J \subseteq I$, $\mathbf{f}|_J \preceq -\sum_{k \in I \smallsetminus J} \mathbf{\Delta}_k|_J$. Note that the fact that $\mathcal{L}$ is a down set has been used to restrict the set of J's to analyse.

A collection of the forbidden pairs $(I, \mathbf{f})$ with smallest $|I|$, such that no other $\mathbf{f}'$ exists with $\mathbf{f}' \succ \mathbf{f}$ and $(I, \mathbf{f}')$ a forbidden pair, in the BTW sandpile with multitoppling rules on all pairs of adjacent sites, is as follows:

$$\begin{array}{|c|}\hline -1 \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline 0 & 0 & 0 \\ \hline\end{array} \qquad \begin{array}{|c|c}\hline 0 & \multicolumn{1}{|c}{} \\ \hline 0 & \multicolumn{1}{c|}{0} \\ \hline\end{array} \quad \dots \quad \begin{array}{|c|c|}\hline 0 & 1 \\ \hline 1 & 0 \\ \hline\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 0 \\ \hline 0 & 1 \\ \hline\end{array} \quad \dots$$

Note that, at difference with the single-site sandpile, in general there is not a unique $\mathbf{f}_{\max}(I)$ such that $(I, \mathbf{f})$ is a forbidden pair if and only if $\mathbf{f} \preceq \mathbf{f}_{\max}(I)$.

At the level of the monoid $\mathcal{M}[a_i, a_i^\dagger]$, we have some extra relations associated to multitoppling rules. For example, consider the action on S'_+. While in the single-site sandpile we have relations (14), in the multitoppling case we also have relations of the form, for each $I \in \mathcal{L}$,

$$\prod_{j \in I} a_j^{(\Delta_I)_j} = \prod_{j \notin I} a_j^{-(\Delta_I)_j} . \tag{28}$$

Note that multiplying left-hand and right-hand sides of relations (14) for all $j \in I$ we would have got a weaker relation, in which the two sides of (28) are multiplied by

$$\prod_{\substack{i \in I \\ j \in I \setminus i}} a_j^{-\Delta_{ij}} , \tag{29}$$

(recall that at the level of the monoid it is not legitimate to take inverses of a_i's, and that, in the equations above, all the exponents are indeed non-negative).

At the level of the equivalence relation $\sim$, and thus of charges $\mathbf{Q}(\mathbf{z})$, nothing changes. In particular, it is easy to see that $\mathbf{u} \sim \mathbf{v}$ in $\mathcal{A}$ if and only if $\mathbf{u} \sim \mathbf{v}$ in $\mathcal{A}_0$. This also leads to the fact that there is exactly one stable recurrent configuration per equivalence class, and that $\oplus$ defines a group structure over R, just as in the single-site toppling sandpile $\mathcal{A}_0 = \mathcal{A}(\mathcal{L}_0)$ associated to the multitoppling sandpile $\mathcal{A} = \mathcal{A}(\mathcal{L})$.

Note however that the set R is *different* from the set R_0 of stable recurrent configurations for $\mathcal{A}_0$, and the fact that they have the same cardinality results from subtle compensations between stable/unstable and recurrent/transient configurations. For example, in the BTW sandpile with multitoppling rules on all pairs of adjacent sites, the set R loses the configurations with adjacent pairs $\boxed{3}\boxed{3}$, which are now unstable, but gains the configurations with adjacent pairs $\boxed{0}\boxed{0}$, which are not FSC's anymore.

A natural bijection between R and R_0, preserving the group structure, is obtained by associating to $\mathbf{z} \in R_0$ the configuration $\mathcal{R}(\mathbf{z}) \in R$, where $\mathcal{R}$ is the complete (multitoppling) relaxation.

In the case of a tight sandpile, the set $R^\dagger$, containing the conjugate of the stable recurrent configurations in the single-toppling sandpile $\mathcal{A}_0$, coincides with the set of recurrent configurations of the sandpile in which $\mathcal{L}$ contains all the subsets of $[n]$. In this case, there are no stable transient configurations, and the condition for $\mathbf{z}$ being stable w.r.t. any toppling I, i.e. $\mathbf{z}|_I \not\succeq \mathbf{\Delta}_I$ for all non-empty $I \subseteq [n]$, is related, by conjugation, to the condition of not having FSC's in $\mathcal{A}_0$, i.e., $(\iota\, \mathbf{z})|_I \not\preceq \mathbf{f}_{\max}(I)$ (because $\mathbf{f}_{\max}(I) = (\overline{\mathbf{z}} - \mathbf{\Delta}_I)|_I$, and, in applying the definition of ι, we should recall that the multitoppling sandpile is formulated in the positive-cone gauge).

Now consider an *anomalous* relaxation process, ρ_I, which may perform a multitoppling rule I only at the first step, if possible, and then perform a single-toppling relaxation with $\mathcal{R}_0$. Such a process is unambiguous, but different processes may not commute, i.e. $\rho_I(\rho_J(\mathbf{z})) \neq \rho_J(\rho_I(\mathbf{z}))$ in general.

Nonetheless, take a whatever semi-infinite sequence $(I_1, I_2, I_3, \ldots)$ of elements in $\mathcal{L} \setminus \mathcal{L}_0$, such that all elements in $\mathcal{L} \setminus \mathcal{L}_0$, occur infinitely-many times. It is easy to see that, for all $\mathbf{z}$, there exists a truncation time $t = t(\mathbf{z})$ such that, for all $s \geq t$, $\rho_{I_s} \rho_{I_{s-1}} \cdots \rho_{I_1}(\mathbf{z}) = \mathcal{R}(\mathbf{z})$, and in particular $\rho_I \mathcal{R}(\mathbf{z}) = \mathcal{R}(\mathbf{z})$ for all $I \in \mathcal{L}$.

The interest in these processes ρ_I is in the fact that their action is strongly related to the action of the idempotents $a_i^\dagger a_i$. Before stating and proving this relation in precise terms, it is instructive to investigate first how this works in the case of the

BTW model. One easily recognizes that $a_i^\dagger a_i \mathbf{z} = \mathbf{z}$ if $z_i < 3$, or $z_i = 3$ and $z_j < 3$ for all the neighbours j of i. In the first case, no topplings or antitopplings are involved, while in the second case exactly one toppling and one antitoppling at i occur. Conversely, if $z_i = 3$ and $z_j = 3$ for some neighbour j, a_i causes an avalanche for which a valid sequence of topplings may start with $(i, j, \dots)$, i.e. $a_i \mathbf{z} = t_{i_k} \cdots t_{i_3} t_j t_i \mathbf{z}$ for some $(i_3, \dots, i_k)$. The *effect* of the two initial topplings is identical to the effect of a multitoppling at a pair $\{i, j\}$. And, crucially, also the *if condition* coincides with the one for a configuration to be unstable w.r.t. the multitoppling at $\{i, j\}$. Thus, a configuration $\mathbf{z} \in \mathbb{Z}^n$ is left stable by the application of $a_i^\dagger a_i$ if and only if it is stable w.r.t. both the toppling i and the multitopplings $\{i, j\}$ for all j neighbours of i. This proves Theorem 2 in the case of the BTW sandpile, and characterizes $\mathbf{y}(\mathbf{z}, I)$ as the result of the relaxation of $\mathbf{z}$, in the multitoppling sandpile for which $\mathcal{L} \smallsetminus \mathcal{L}_0 = \big\{\{i,j\}\big\}_{i \,\in\, I,\; d(i,j)\,=\,1}$.

The proof in the general setting, that we present below, is completely analogous.

Proof of Theorem 2. One finds that $a_i^\dagger a_i \mathbf{z} = \mathbf{z}$ if $z_i < \overline{z}_i$, or $z_i = \overline{z}_i$ and $z_j \leq \overline{z}_j + \Delta_{ij}$ for all $j \neq i$. Again, in the first case no topplings or antitopplings are involved, while in the second case exactly one toppling and one antitoppling at i occur.

Conversely, if the conditions above are violated, a_i causes an avalanche for which a valid sequence of topplings may start with $(i, j, \dots)$, and the effect of the two initial topplings is identical to the effect of a multitoppling at a pair $\{i, j\}$.

The if condition for the multitoppling $\{i, j\}$ to occur is that $z_i > \overline{z}_i + \Delta_{ji}$ and $z_j > \overline{z}_j + \Delta_{ij}$. The condition for the avalanche to involve topplings at i and j is $z_i = \overline{z}_i$ and $z_j > \overline{z}_j + \Delta_{ij}$. These two sets of conditions are certainly simultaneously not satisfied if $\Delta_{ij} = 0$, thus we can restrict our attention to sites j such that $\Delta_{ij} < 0$. In this case, the two sets coincide if and only if $\Delta_{ji} = -1$. Thus, in order to make the sets coincide for all sites i, we need that Δ is symmetric, and all the non-zero off-diagonal entries are -1, as required in the theorem hypotheses.

The configuration $\mathbf{y}(\mathbf{z}, I)$ is the result of the relaxation of $\mathbf{z}$, in the multitoppling sandpile for which $\mathcal{L} \smallsetminus \mathcal{L}_0 = \big\{\{i,j\}\big\}_{i \in I, \Delta_{ij} = -1}$. □

4 Discussion of Markov Chain dynamics involving both sand addition and removal

In this section we discuss several dynamics involving the operators a_i and $a_i^\dagger$. Each example has a different theoretical and phenomenological motivation, and is intended to describe a different feature of out-of-equilibrium steady states. To keep the visualisation simple, all our examples are variations of the BTW model, on portions of the square lattice and with heights in the range $\{0, 1, 2, 3\}$.

4.1 A reminder of the Karmakar and Manna protocol

The first and most natural dynamics in this family is the modification of the BTW sandpile, where, with probability p and $1 - p$ an a_i or an $a_i^\dagger$ move is performed, at a randomly chosen site i. This dynamics has been investigated in detail by Karmakar and Manna in [31]. For $p > 1/2$ and $p < 1/2$, the system has features resembling the ones of the ordinary BTW model, and of its symmetric image under the involution $z_i \leftrightarrow 3 - z_i$. For example, for $p > 1/2$ the avalanches have an algebraic-tail distribution (the exponent varies with p), while the anti-avalanches have a distribution with most of the support on values of order 1. Of course, these two features are swapped for $p < 1/2$. In a window near $p = p_c = 1/2$, for which several scaling exponents have

been investigated numerically, aspects of a new criticality phenomenon emerge. For example, at $p = p_c$ the distribution of sizes of both avalanches and anti-avalanches seem to follow a stretched-exponential.

As another example of new emerging features, both the variance and the correlation times of the total mass in the system increase when p_c is approached, and the variation in p of the average total mass diverges at p_c, with a new scaling exponent.

4.2 Dynamics preserving the total mass

Our first dynamics is a minor modification of the one described above, in the microcanonical ensemble with fixed total mass, at its critical value. Among other things, this allows to use periodic boundary conditions. Indeed, the dynamics studied by Karmakar and Manna, even at $p = p_c$, if implemented on a graph with no boundary, may produce an infinite avalanche or anti-avalanche, while in a dynamics with conserved mass, with randomized initialisation, if the initial density is in the range $3 - \rho_* < \rho < \rho_*$, where ρ_* is the critical density (having value $\rho^* = 2.125288\ldots$ [41,42], note that, for a long time, it was mistakenly believed that $\rho^* = 17/8$), in the large volume limit we are protected from infinite avalanches.

We want to emphasise here the behaviour of thermalisation process at short times, that shows coarsening, and an aggregation phenomenon, which is purely dynamical.

More generally, under some aspects, our dynamics compares to the one of Karmakar and Manna roughly as Ising at fixed magnetisation under Kawasaki dynamics relates to Ising heat-bath evolution. Recall however that the presence of conserved quantities has more dramatic effects in non-equilibrium systems than it has in Boltzmann equilibrium theory (see e.g. the very different features of the Driven Lattice Gas, at conserved number of particles or with open boundaries [44]).

The precise time evolution is as follows. At each time, one of the $2n^2$ operators $\{a_j^\dagger a_i, a_i a_j^\dagger\}$ is chosen uniformly at random, and applied to the previous state. For the examples shown in Fig. 1, we have chosen as initialisation the one best approximating a uniform distribution of the mass, $z_{(x,y)} = 1$ if $x + y$ is odd and 2 if it is even. As seen from the figure, the coarsening is already evident at the level of the height variables. It is however much more evident at the level of a local parameter suggested by the Burning Test. The ordinary Burning Test cannot be performed here, as we have no boundary. We choose as *effective boundary* the set of sites with maximal height. More precisely, we record the sites that perform at least one toppling, in the avalanche following the replacement $3 \to 4$, and colour them in blue. Similarly, we record the sites that perform at least one anti-toppling, in the avalanche following the replacement $0 \to -1$, and colour them in red. In case both a toppling and an antitoppling have occurred, we break the tie by observing the initial height. In case neither a toppling nor an antitoppling have occurred, we use a *neutral* light-yellow colour. Thus, pictorially, blue, red and yellow regions correspond to regions which are recurrent, anti-recurrent, and simultaneously transient and anti-transient. The resulting analysis of the configurations is shown in Fig. 2.

4.3 Dynamics with idempotent operators

Our second dynamics starts from the maximally filled configuration, $z_i = 3$ for all i, and acts with the idempotent combinations $a_i^\dagger a_i$ at randomly-chosen sites.

As we know from Theorem 2, this dynamics is absorbent on a unique configuration, identified with the multitoppling relaxation of the initial configuration, where pairs of adjacent sites both with height 3 are unstable.

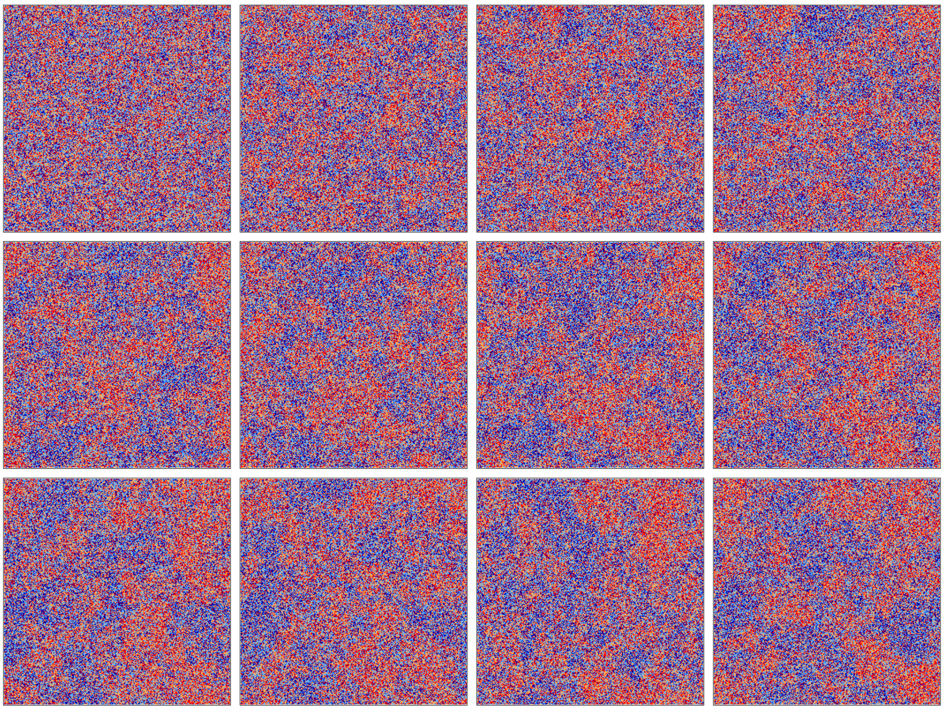


Fig. 1. Time evolution of the Markov Chain described in Section 4.2. Here $L = 256$, and times shown are $t = L^2, 4L^2, 9L^2, \ldots, 144L^2$. Color code: $(0, 1, 2, 3) = (\text{red}, \text{orange}, \text{cyan}, \text{blue})$.

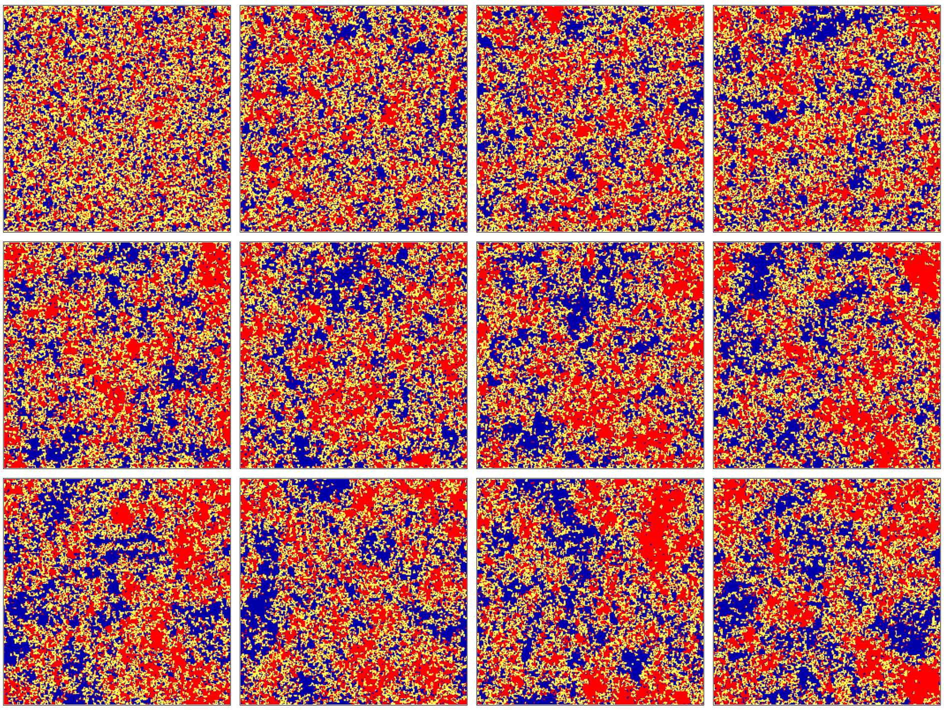


Fig. 2. The local parameter associated to burning-test defined in the text of configurations in Fig. 1.

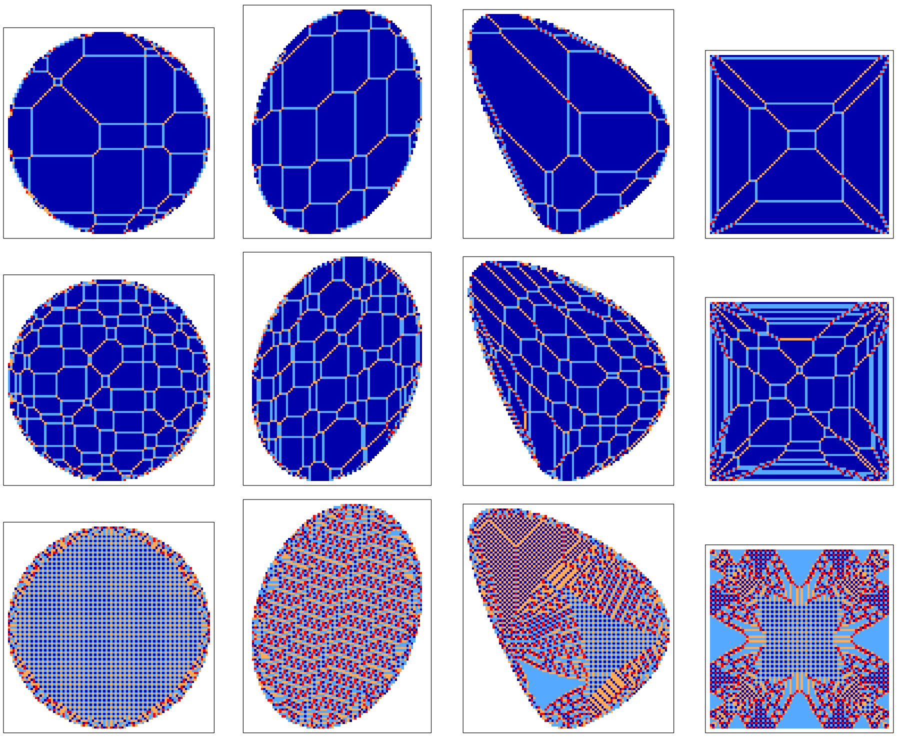


Fig. 3. Configurations obtained with the procedure described in Section 4.3. Top, middle and bottom row correspond to $t = 32$, $t = 1024$, and to the fixed point of the dynamics. The columns show different domains. From left to right: a circle; an ellipse with axes rotated by $\arctan 1/3$ w.r.t. the cartesian axes, and ratio 2 between height and width; a typical algebraic curve of degree 3, more precisely $4x^2 + 4y^2 + 3xy + 2x^2y + x^3 = 7$; a square.

Again, interesting features emerge at short times, when the configuration takes the form of a *web of strings*, satisfying a classification theorem and a collection of incidence rules [27].

On the other extremum of the dynamics, at the fixed point we have configurations showing remarkable regularities, in the form of *patches*, that is, a local two-dimensional periodicity on portions of the domain [25]. When more patches are present, they follow an incidence rule [26].

If the initial domain is an elliptic portion of the square lattice, a specially higher regularity emerges. Say that the linear dimension of the domain is of order L, and the slope of the symmetry axes is a *small* rational p/q (with both p and q of $\mathcal{O}(1)$ in L). Then, in the limit $L \to \infty$, we observe the emergence of a very simple structure of patches and strings: we have a unique patch, crossed by strings of a unique type, parallel to one of the two symmetry axes. This fact is in agreement with the general theory developed in [25,26,27], as the toppling vector at a coarsened level is a quadratic form in the coordinates x and y, that should vanish at the boundary of the domain, and the contour lines of quadratic forms are conics, i.e. plane algebraic curves of degree 2.

In Fig. 3 we present configurations obtained with the procedure described above, starting with the maximally-filled configuration, $z_i = 3$ for all i, on portions of the square lattice of various shapes, in order to highlight the features outlined above: a

disk, an ellipse, a smooth domain which is not a conic (it is an algebraic curve of degree 3), and a square.

4.4 A simple deterministic dynamics

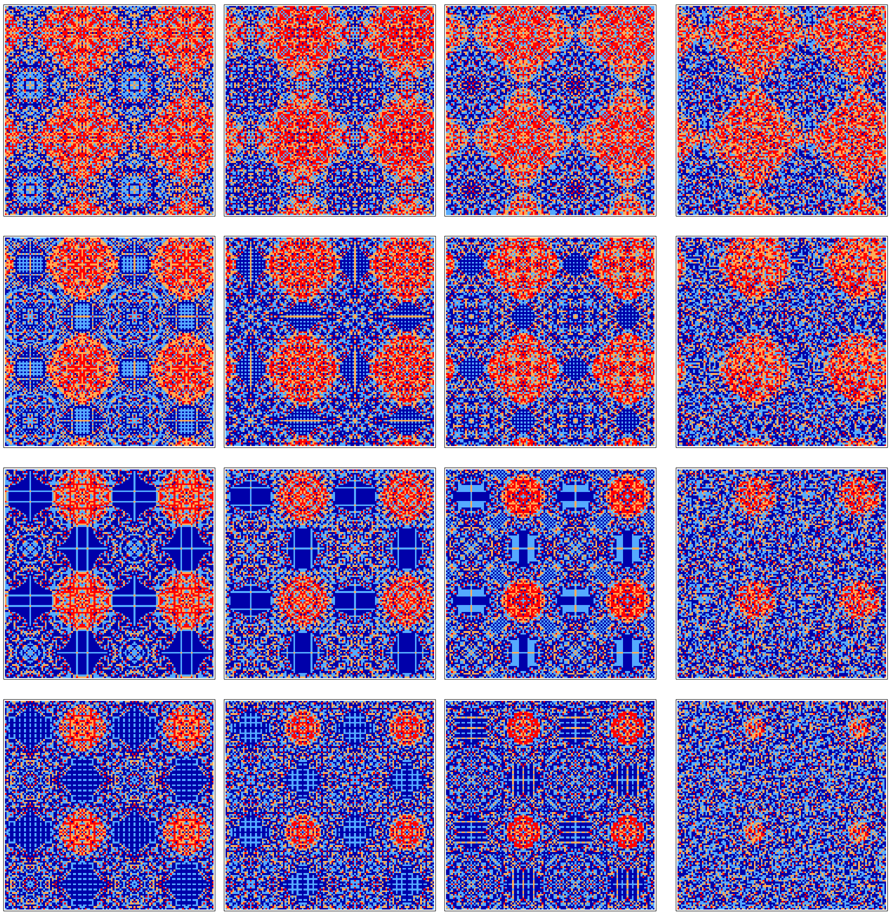


Fig. 4. Examples of configurations, for $L = 60$, observed in the dynamics described in Section 4.4. The domain is repeated periodically 2×2 times, in order to improve visualisation. The different rows correspond to densities $\rho = 1.5,\ 1.75,\ 2,\ 2.111\ldots$. The three left-most columns show, respectively, configurations at time $t = 3000$, 6000 and 9000, obtained by starting from periodic configurations prepared by using the tiles $\left(\begin{smallmatrix}2&1\\1&2\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2&2\\2&1\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}2\end{smallmatrix}\right)$, and $\left(\begin{smallmatrix}3&2&2\\2&2&2\\2&2&2\end{smallmatrix}\right)$. The last column shows, for comparison, the configurations at $t = 3000$ obtained by starting from random configurations at the given densities.

Our third and last example of dynamics is deterministic, and corresponds to the sand flow from a unique source site i to a unique sink site j in the lattice, through the iterated application of the operator $a_j^\dagger a_i$.

Besides producing beautiful drawings, this dynamics has an interesting phenomenological feature. Contrarily to the naïve intuition of a smooth hydrodynamic flow from

the source to the sink, in typical configurations at the steady state, the density profile presents a *shock*, between a spatial region analogous to ordinary BTW and a region analogous to the involution image of the BTW, both regions presenting large avalanches (of the appropriate type).

This phenomenology seems an analogue of the shock profiles observed in various non-equilibrium systems, most notably a two-dimensional analogue of a feature of the TASEP [45], an exactly-solvable model in one dimension, in the case in which two species of particles are considered [46].

We note that, in comparison with the previous realisations, this dynamics presents short thermalisation times and very moderate aging. (both the thermalisation time and the aging effect seem to become more relevant as ρ approaches ρ^*). A partial justification is the fact that this dynamics is deterministic and thus, after a transient, it follows a periodic orbit, although the growth of the period with the system size is extremely fast. Recall that the length of the orbit of the operator $a_j^{-1} a_i$, acting in the torus $G \cong \mathbb{Z}_{d_1} \times \mathbb{Z}_{d_2} \times \cdots \times \mathbb{Z}_{d_g}$ of recurrent configurations, for the abelian version of the model, is a divisor of the length of the orbit of $a_j^\dagger a_i$ acting on S, due to the fact that the equivalence classes are preserved in the framework with both topplings and antitopplings.

We presents configurations realised on a $L \times L$ torus, with sink and source at antipodal points, i.e. acting with the operator $a_j^\dagger a_i = a_{(L/2,L/2)}^\dagger a_{(0,0)}$, which is the simplest and most symmetric realisation of the ideas above. We studied both the case in which the initial state is periodic, and the case of random initialisation with given prescribed density. The configurations are presented in Fig. 4.

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